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STABILIZATION OF SOLUTIONS OF THE NONLINEAR EQUATION OF FILTRATION OF A TWO-PHASE LIQUID

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The asymptotic behavior (with unlimited increase in time) of solutions of boundary-value problems for the filtration equation for a two-phase liquid that describe the displacement of immiscible incompressible liquids from a bed is studied. Convergence of these solutions to the unique solution of the steady problem (stabilization) is established, and, under additional assumptions, the rate of convergence is evaluated.

The initial boundary-value problem for the dynamic saturation $s(x,t)$ in the region $\Omega\{0 \leqslant x \leqslant l, 0 \leqslant t < \infty\}$ has the form [1]

$$\text{æ}(x)\frac{\partial s}{\partial t} = \frac{\partial}{\partial x}\Big(k(x)a(s)\frac{\partial s}{\partial x} - Q(t)b(s)\Big), \tag{1}$$

$$s(0,t) = s_1(t), \quad s(l,t) = 0, \quad s(x,0) = s_0(x), \quad 0 \leqslant s_1(t) \leqslant 1, \quad 0 \leqslant s_0(x) \leqslant 1, \tag{2}$$

where $\text{æ}(x) > 0$ and $k(x) > 0$ are the ground porosity and the filtration coefficient, respectively, and $Q(t) > 0$ is the total liquid flow.

The functions $a(s)$ and $b(s)$ are defined for $0 \leqslant s \leqslant 1$ and satisfy the conditions

$$\begin{gathered} a(s) > 0 \quad \text{for} \quad 0 < s < 1, \quad a(0) = a(1) = 0, \\ b(s) > 0, \quad b'(s) > 0 \quad \text{for} \quad 0 < s < 1, \quad b'(0) \geqslant 0, \quad b'(1) \geqslant 0. \end{gathered} \tag{3}$$

Thus, the parabolic equation (1) degenerates into a first-order equation for the following two values of the required function: $s = 0$ and $s = 1$.

Remark. The change of variables

$$t = t, \qquad \xi = \int\limits_0^x \frac{d\tau}{k(\tau)\varphi(1)} \equiv \chi(x), \qquad \varphi(s) = \int\limits_0^s a(\tau)\,d\tau$$

transforms Eq. (1) to the equation

$$\nu(\xi)\frac{\partial s}{\partial t} = \frac{\partial}{\partial \xi}\Big(\frac{a(s)}{\varphi(1)}\frac{\partial s}{\partial \xi} - Q(t)b(s)\Big),$$

where $\nu(\xi) = k(x(\xi))\text{æ}(x(\xi))\varphi(1)$ and $x(\xi) \equiv \chi^{-1}(\xi)$ is a function that is inverse for $\xi = \chi(x)$. Hence, without loss of generality, we can set $k(x) \equiv 1$, $\text{æ}(x) \equiv \nu(x)$, and $\varphi(1) = 1$.

S. N. Antontsev and V. N. Monakhov pioneered investigation into the correctness of nonlinear boundary-value filtration problems for a two-phase liquid. Antontsev and Kazhikohov [1] formulated conditions for the weak convergence of solutions of the boundary-value problems to steady solutions.

Khusnutdinova [2, 3] and Artemova and Khusnutdinova [4] studied the stabilization of solutions of the boundary-value problems for the nonlinear equation of single-phase filtration.

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In [5] (see also [6]) it is proved that a generalized solution $s(x,t) \in C(\Omega_T)$ of problem (1), (2) that satisfies the inequalities $0 \leqslant s(x,t) \leqslant 1$ and Eq. (1) in the sense of integral identity exists in the region $\Omega_T\{0 \leqslant x \leqslant l, 0 \leqslant t \leqslant T\}$ for any $T > 0$. It is also established that $|\partial\varphi(s)/\partial x| \leqslant M$, $(x,t) \in \Omega_T$ with the constant M independent of T, which guarantees, in particular, finiteness of the phase flows $v_1 = -(\partial\varphi/\partial x - Qb)$ and $v_2 = Q - v_1$.

As in the case $s_1(t) \equiv 1$ and $\nu(x) \equiv 1$ (see [5]), the existence of such a generalized solution for boundary-value problem (1), (2) in the region Ω is proved. In this case, the corresponding integral identity has the form

$$\iint\limits_{\Omega} \left[\nu s \frac{\partial f}{\partial t} - \frac{\partial\varphi}{\partial x}\frac{\partial f}{\partial x} + Q(t)b(s)\frac{\partial f}{\partial x}\right] dt\,dx + \int_0^l \nu(x)s_0(x)f(x,0)\,dx = 0, \tag{4}$$

where $f(x,t) \in C^1(\Omega)$ is any function which is equal to zero at $x = 0$ and $x = l$, and outside the finite region.

Sufficient conditions for the existence of a generalized solution to problem (1), (2) in the region Ω are

$$\text{(i)}\quad s_0(x) \in C[0,l],\quad \varphi[s_0(x)] \in C^1[0,l],\quad \nu(x) \in C^{1+\alpha}[0,l],\ s_1(t) \in C^2[0,\infty),$$
$$a(s) \in C^{1+\alpha}[0,1],\quad b(s) \in C^{2+\alpha}[0,1],\quad Q(t) \in C^{(1+\alpha)/2}[0,\infty),\quad \alpha \in (0,1),$$
$$s_1'(t) > 0,\quad 0 < Q(t) \leqslant Q_0,\quad t \geqslant 0,\quad 0 = s_0(l) \leqslant s_0(x) \leqslant 1,\quad s_0(0) = s_1(0).$$

The function $\sigma(x) \in C[0,l]$ satisfying the inequalities $0 \leqslant \sigma(x) \leqslant 1$ will be called a generalized solution of the boundary-value problem

$$\frac{d}{dx}\left[a(\sigma)\frac{d\sigma}{dx} - Q_0 b(\sigma)\right] = 0, \qquad Q_0 = \max Q(t) > 0, \tag{5}$$

$$\sigma(0) = 1, \qquad \sigma(l) = 0, \tag{6}$$

if there is a limited generalized derivative $d\varphi(\sigma)/dx$ and the following integral identity of the form (4) holds:

$$\int_0^l \left[a(\sigma)\frac{d\sigma}{dx} - Q_0 b(\sigma)\right]\frac{df}{dx}\,dx = 0. \tag{7}$$

Here $f(x) \in C^1[0,l]$ is an arbitrary function which is equal to zero at $x = 0$ and $x = l$.

Let us make additional assumptions:

$$0 \leqslant Q_0 - Q(t) \leqslant M_1(t+1)^{-\gamma_1}, \qquad 0 \leqslant 1 - s_1(t) \leqslant M_2(t+1)^{-\gamma_1}, \qquad \gamma_1 > 0,$$
$$0 < s_0(x) \leqslant \sigma(x), \qquad x \in [0,l), \qquad \lim_{x\to l} a(s_0)s_0'(x) < 0, \qquad s_1(0) > 0.$$

By virtue of the monotonicity of the function $\varphi(s) = \int_0^s a(\tau)\,d\tau$, the above assumptions are obviously equivalent to

$$0 \leqslant 1 - \varphi[s_1(t)] \leqslant M_3(t+1)^{-\gamma_1}, \qquad 0 \leqslant Q_0 - Q(t) \leqslant M_1(t+1)^{-\gamma_1}; \tag{8}$$

$$\varphi(k_1)\frac{l-x}{l} \leqslant \varphi[s_0(x)] \leqslant \varphi[\sigma(x)], \tag{9}$$

where M_3, M_1, γ_1, and $k_1 \leqslant s_1(0)$ are certain constants.

Theorem. *If conditions* (i) (3), (8), and (9) are satisfied, *the generalized solution of problem* (1), (2) *in the limit $t \to \infty$ tends to the unique generalized solution $\sigma(x)$ of problem* (5), (6), *which satisfies the inequalities*

$$0 \leqslant \sigma(x) \leqslant 1, \quad \frac{d\sigma}{dx} \leqslant 0, \quad \frac{d^2\sigma}{dx^2} \leqslant 0, \quad x \in [0,l], \tag{10}$$

and, if

$$b'(\sigma) \geqslant b_0 = \text{const} > 0, \qquad 0 \leqslant \sigma \leqslant 1, \tag{11}$$

the estimate

$$|\varphi[s(x,t)] - \varphi[\sigma(x)]| \leqslant \frac{M}{(t+1)^\gamma} \qquad [(x,t) \in \Omega], \tag{12}$$

holds, where M and $\gamma > 0$ depend on M_3, M_1, b_0, γ_1, l, Q_0, and the other data of the problem.

Proof. By changing the required functions $\varphi(s) = v$ and $\varphi(\sigma) = u$, we brought boundary-value problems (1), (2) and (5), (6) to the form

$$L(v) \equiv A(v)\,\frac{\partial^2 v}{\partial x^2} - Q(t)B(v)\,\frac{\partial v}{\partial x} - \nu(x)\,\frac{\partial v}{\partial t} = 0, \tag{13}$$

$$v(0,t) = v_1(t), \qquad v(x,0) = v_0(x), \qquad v(l,t) = 0, \tag{14}$$

$$A(u)\,\frac{d^2 u}{dx^2} - Q_0 B(u)\,\frac{du}{dx} = 0, \tag{15}$$

$$u(0) = \varphi(1) = 1, \qquad u(l) = \varphi(0) = 0. \tag{16}$$

Here $A(v) \equiv a(\Phi(v))$, $B(v) \equiv b'(\Phi(v))$, $\Phi(v) \equiv \varphi^{-1}(v) = s$, $v_1(t) = \varphi[s_1(t)]$, $v_0(x) = \varphi[s_0(x)]$, and $v(l,t) = \varphi[s(l,t)] = 0$.

Since $\varphi'_s = a(s) > 0$ for $0 < s < 1$, the s and v are in one-to-one correspondence.

As in [5], a solution $s(x,t)$ of boundary-value problems (1), (2) is obtained in the limit $n \to \infty$ and $m \to \infty$ of the classical solutions $s_{mn}(x,t) \equiv \Phi[v_{mn}(x,t)]$ of Eq. (1), where $v_{mn}(x,t) \in C^{3+\alpha}(\Omega)$ satisfy Eq. (13), the regularized initial boundary conditions

$$v_{mn}(x,0) = v_{0mn}(x), \qquad v_{mn}(0,t) = v_{1mn}(t) \equiv v_1(t) - 1/m, \qquad v_{mn}(l,t) = 1/n, \tag{17}$$

$$\lim_{m\to\infty}(\lim_{n\to\infty} v_{0mn}(x)) = v_0(x), \tag{18}$$

and the inequalities

$$\frac{1}{n} \leqslant v_{mn}(x,t) \leqslant 1 - \frac{1}{m}, \qquad \left|\frac{\partial v_{mn}}{\partial x}\right| \leqslant M. \tag{19}$$

By construction (see [5]), $v_{mn}(x,t)$ form sequences monotonically decreasing in n and monotonically increasing in m. The same is also true for $s_{mn}(x,t) \equiv \Phi[v_{mn}(x,t)]$.

Similarly, a solution $\sigma(x)$ of problem (5) (6) can be represented as the limit of the sequence of the functions $\sigma_{mn}(x) \equiv \Phi[u_{mn}(x)]$ as $n \to \infty$ and $m \to \infty$, where $u_{mn}(x) \in C^{3+\alpha}[0,l]$ are solutions of Eq. (15) that satisfy the boundary conditions [cf. (16)]

$$u_{mn}(0) = 1 - 1/m, \qquad u_{mn}(l) = 1/n \tag{20}$$

and are such that

$$\frac{1}{n} \leqslant u_{mn}(x) \leqslant 1 - \frac{1}{m}, \qquad \frac{du_{mn}}{dx} < 0, \qquad \frac{d^2 u_{mn}}{dx^2} < 0, \qquad x \in [0,l]. \tag{21}$$

We first prove that a solution $u_{mn}(x)$ of problem (15), (20) that satisfies inequalities (21) exists and is unique.

For simplicity, omitting the subscripts at u_{mn} and writing Eq. (5) as $(d/dx)[du/dx - Q_0 b(\Phi(u))] = 0$, we obtain

$$\frac{du}{dx} = Q_0 b[\Phi(u)] - Q_0 C. \tag{22}$$

Integrating once again, we come to the following implicit representation of the solution:

$$x = l - \frac{1}{Q_0} \int\limits_{1/n}^{u} \frac{d\tau}{C - b[\Phi(\tau)]} \equiv l - \frac{1}{Q_0} f(C, u), \tag{23}$$

where C is an arbitrary constant. For $C = b[\Phi(1 - 1/m)]$, obviously, $lQ_0 - f(b[\Phi(1-1/m)], 1 - 1/m) < 0$. In addition, $f(C, 1 - 1/m) \to 0$ as $C \to \infty$ and, hence, $lQ_0 - f(C, 1 - 1/m) > 0$ for rather large C. Hence, for a certain $C = C_0 > b[\Phi(1 - 1/m)]$, we have $f(C_0, 1 - 1/m) = lQ_0$, i.e.,

$$x(C_0, 1 - 1/m) = 0, \qquad x(C_0, 1/n) = l. \tag{24}$$

Since $C_0 > b[\Phi(1 - 1/m)]$ and $x'_u < 0$, from Eqs. (22)–(24) it follows that there is a unique solution of problem (15), (20) that possesses the properties (21), and, thus, there is a unique solution of Eq. (5) that satisfies the boundary conditions

$$\sigma_{mn}(0) = \Phi(1 - 1/m), \qquad \sigma_{mn}(l) = \Phi(1/n) \tag{25}$$

and the inequalities

$$\Phi(1/n) \leqslant \sigma_{mn}(x) \leqslant \Phi(1 - 1/m), \quad \frac{d\sigma_{mn}}{dx} < 0, \quad \frac{d^2\sigma_{mn}}{dx^2} < 0, \quad x \in [0, l]. \tag{26}$$

Let us compare the functions $u_{mn}(x)$ and $u_{m(n+1)}(x)$ on the segment $[0, l]$. At $x = 0$, we have $u_{mn}(0) = u_{m(n+1)}(0)$; at $x = l$, we obtain $u_{mn}(l) > u_{m(n+1)}(l)$.

In the interval $0 < x < l$, the difference $z = u_{mn}(x) - u_{m(n+1)}(x)$ satisfies the linear parabolic equation

$$A(u_{mn}) \frac{d^2 z}{dx^2} - Q_0 B(u_{mn}) \frac{dz}{dx} + Rz = 0, \tag{27}$$

where $R = A'_u(\theta_1) \dfrac{d^2 u_{m(n+1)}}{dx^2} - Q_0 B'_u(\theta_2) \dfrac{du_{m(n+1)}}{dx}$ and θ_1 and θ_2 are values that are intermediate between $u_{mn}(x)$ and $u_{m(n+1)}(x)$.

By virtue of (27), the function $w = z(x)\,\mathrm{e}^{-\beta t}$ is a solution of the equation

$$A(u_{mn}) \frac{\partial^2 w}{\partial x^2} - Q_0 B(u_{mn}) \frac{\partial w}{\partial x} + (R - \beta) w - \frac{\partial w}{\partial t} = 0.$$

We choose β from the condition $\beta - R > 0$, $x \in (0, l)$. This is possible because of the boundedness of R. Hence, according to the maximum principle, we conclude that $w(x, t) \geqslant 0$ at $x \in [0, l]$, i.e., $u_{mn}(x) \geqslant u_{m(n+1)}(x)$. Similarly we see that $u_{mn}(x) \leqslant u_{(m+1)n}(x)$, $x \in [0, l]$.

Thus, the sequence $\{u_{mn}(x)\}$ is monotonically decreasing in n and monotonically increasing in m. The same is true for the sequence of the functions $\sigma_{mn}(x) \equiv \Phi[u_{mn}(x)]$, where $0 < \sigma_{mn}(x) < 1$ because of the continuity of Φ.

By virtue of these properties, $\lim\limits_{n \to \infty} \sigma_{mn}(x) = \sigma_m(x)$ exists for every m. Obviously, $0 \leqslant \sigma_m(x) \leqslant \sigma_{m+1}(x) < 1$, $x \in [0, l]$, and, hence, $\lim\limits_{m \to \infty} \sigma_m(x) = \sigma(x)$ exists.

With allowance for (21), from (22) we obtain

$$\left| \frac{d\varphi(\sigma_{mn})}{dx} \right| \leqslant M_0,$$

where M_0 does not depend on m and n. This implies that the limiting function $\varphi[\sigma(x)]$ satisfies the Lipschitz condition and in Ω there is a generalized derivative $d\varphi(\sigma)/dx$, which does not exceed M_0 in absolute value and is a $*$-weak limit of a certain subsequence $d\varphi[\sigma_{m_k n_k}]/dx$ ($\{\sigma_{m_k n_k}\} \subset \{\sigma_{mn}\}$). Passing to the limit $n_k \to \infty$ and $m_k \to \infty$ for $\sigma_{m_k n_k}(x)$ in identity (7), we prove the validity of (7) for $\sigma(x)$.

Thus, $\sigma(x)$ is a generalized solution of boundary-value problem (5), (6) that satisfies (10) [see (25) and (26)]. We show that the solution of problem (5), (6) that possesses the properties (10) is unique.

In the case $lQ_0 < \int\limits_0^1 \frac{d\tau}{b(1) - b[\Phi(\tau)]}$, this follows from the following equality [see (23) in the limit $n \to \infty$ and $m \to \infty$]:

$$x = l - \frac{1}{Q_0} \int\limits_0^u \frac{d\tau}{C_0 - b[\Phi(\tau)]} \equiv l - \frac{1}{Q_0} f(C_0, u) \equiv F(C_0, u), \tag{28}$$

which is invertible by virtue of the positiveness of $C_0 - b(1)$, $f(C_0, 0) = 0$, and $f(C_0, 1) = lQ_0$, i.e., $\sigma(x) = \Phi[F^{-1}(x)]$ is uniquely determined from (28).

If $\lim\limits_{C_0 \to b(1)} F(C_0, 1) < \infty$ and $lQ_0 \geqslant f[b(1), 1] = l_1 Q_0$, then, from (22) we obtain

$$\left.\frac{du}{dx}\right|_{x=l-l_1} = \left.\frac{d\varphi(\sigma)}{dx}\right|_{x=l-l_1} = 0.$$

Taking the latter into account and assuming for $l > l_1$ that

$$\sigma(x) = \begin{cases} 1, & x \in [0, l - l_1], \\ \Phi[F^{-1}(x)], & x \in (l - l_1, l], \end{cases}$$

we see that $\sigma(x)$ is a nonincreasing, upward convex, generalized solution of problem (5), (6). By construction, this solution is obviously unique.

To complete the proof of the theorem, we check the validity of the inequalities

$$u_{mn}[\psi(x,t)] \leqslant v_{mn}(x,t) \leqslant u_{mn}(x), \tag{29}$$

where $\psi = (x - l)(t+1)^\gamma / ((t+1)^\gamma + \delta) + l$, $(m, n) > N_0$, $\delta > \delta_0 > 1$, and $\gamma < \gamma_0 < \gamma_1$ and N_0, δ_0, and γ_0 are some constants.

By virtue of conditions (8), (9), (14), (17)–(20), inequalities (29) are satisfied for rather large N_0, δ_0, and γ_0^{-1} on Γ (a portion of the boundary of the rectangle Ω that consists of the lateral sides $x = 0$ and $x = l$ and the lower base $t = 0$).

Thus, for any $n > N_0$ and $m > N_0$, the difference $q = u_{mn}(x) - v_{mn}(x,t)$ is nonnegative on Γ and in the region Ω (the subscripts at u_{mn} and v_{mn} are omitted) it satisfies the linear parabolic equation

$$L_1(q) \equiv A(v) \frac{\partial^2 q}{\partial x^2} - Q(t) B(v) \frac{\partial q}{\partial x} - c_1 q - \nu \frac{\partial q}{\partial t} = f_1,$$

where $f_1 = [Q_0 - Q(t)] B(u) (du/dx) \leqslant 0$ and the coefficient $c_1 = A'_v(\theta_1)(d^2u/dx^2) - Q(t) B'_v(\theta_2)\,(du/dx)$ (where θ_1 and θ_2 are intermediate values between v_{mn} and u_{mn}) is limited in Ω by a certain constant M_4 [see (8) and (22)].

Converting to the function $\chi = q\mathrm{e}^{-\beta t}$ ($\nu\beta > M_4$), we obtain

$$\left.\chi\right|_\Gamma \geqslant 0, \quad \mathrm{e}^{-\beta t} L_1(\chi \mathrm{e}^{\beta t}) \equiv L_1(\chi) - \nu\beta\chi = f_1 \mathrm{e}^{-\beta t} \leqslant 0, \quad (x,t) \in \Omega.$$

Hence, according to the maximum principle, $\chi = q\mathrm{e}^{-\beta t} \geqslant 0$ for $(x,t) \in \Omega$, i.e., $v_{mn}(x,t) \leqslant u_{mn}(x)$, $(x,t) \in \Omega$.

Let us prove the left side of inequalities (29). We set $\mu = (t+1)^\gamma$,

$$\tau = u_{mn}(\psi), \quad \psi = \psi(x,t), \quad L_0(\tau) \equiv A(\tau) \frac{\partial^2 \tau}{\partial x^2} - Q_0 B(\tau) \frac{\partial \tau}{\partial x} - \nu \frac{\partial \tau}{\partial t}$$

and calculate $L_0(\tau)$, $L_2(v - \tau) \equiv L(v) - L_0(\tau) - d$, where

$$d = [Q_0 - Q(t)] B(\tau) \frac{d\tau}{d\psi} \frac{\mu}{\mu + \delta}, \quad L_2(z) = A(v) \frac{\partial^2 z}{\partial x^2} - Q(t) B(v) \frac{\partial z}{\partial x} + c_2 z - \nu \frac{\partial z}{\partial t},$$

$$c_2 = A'_v(\theta_1) \frac{\partial^2 \tau}{\partial x^2} - Q(t) B'_v(\theta_2) \frac{\partial \tau}{\partial x}$$

(θ_1 and θ_2 are intermediate values between v_{mn} and τ). We have

$$L_0(\tau) = \left(\frac{\mu}{\mu+\delta}\right)^2\left[A(\tau)\frac{d^2\tau}{d\psi^2} - Q_0B(\tau)\frac{d\tau}{d\psi}\right] + Q_0B(\tau)\frac{d\tau}{d\psi}\frac{\mu}{\mu+\delta}\left(\frac{\mu}{\mu+\delta}-1\right)$$
$$-\frac{\gamma\nu\delta\mu}{(t+1)(\mu+\delta)^2}\frac{d\tau}{d\psi}(x-l) = \frac{\mu\delta}{(\mu+\delta)^2}\left|\frac{d\tau}{d\psi}\right|\left[Q_0B(\tau) + \frac{\gamma\nu(x-l)}{t+1}\right],$$
$$L_2(v-\tau) = -\frac{\mu\delta}{(\mu+\delta)^2}\left|\frac{d\tau}{d\psi}\right|\left[Q_0B(\tau)+\frac{\gamma\nu(x-l)}{t+1}\right] + [Q_0 - Q(t)]B(\tau)\left|\frac{d\tau}{d\psi}\right|\frac{\mu}{\mu+\delta}$$
$$\leqslant -\frac{\mu}{\mu+\delta}\left|\frac{d\tau}{d\psi}\right|\left\{\left[\frac{Q_0\delta}{(t+1)^\gamma+\delta} - \frac{M_1}{(t+1)^{\gamma_1}}\right]B(\tau) - \frac{\gamma\nu l\delta}{(t+1)(\mu+\delta)}\right\} \equiv \lambda_0.$$

Since $\lambda(t) = Q_0\delta/((t+1)^\gamma+\delta) - M_1/(t+1)^{\gamma_1} > 0$ for $\gamma < \gamma_1/2$ and $t \geqslant t_0$ (t_0 is a certain integer), without loss of generality, we can assume that $t_0 = 0$, i.e., $\lambda(0) > 0$.

Next, taking into account that $B(\tau) \equiv b_s'[\sigma_{mn}[\psi)] \geqslant k_0(m,n)$ [$k_0 \to 0$ as $n \to \infty$ and $m \to \infty$, see (3)] and choosing γ from the condition

$$\gamma = \frac{1}{2}\min\left\{\frac{\lambda(0)k_0}{\nu l}, \gamma_1\right\} = \gamma_0,$$

we find that $L_2(v-\tau) \leqslant \lambda_0 < 0$ at $(x,t) \in \Omega$.

Thus, the function $\omega = (v-\tau)\,\mathrm{e}^{-\beta t}$ ($\nu\beta > |c_2|$) satisfies the conditions

$$\omega\Big|_\Gamma \geqslant 0, \quad \mathrm{e}^{-\beta t}L_2(\omega\,\mathrm{e}^{\beta t}) = L_2(\omega) - \nu\beta\omega = \lambda_0\mathrm{e}^{-\beta t} < 0, \quad (x,t) \in \Omega,$$

which ensure, according to the maximum principle, that $\omega = (v-\tau)\,\mathrm{e}^{-\beta t}$ is nonnegative everywhere in Ω, i.e., $v_{mn}(x,t) \geqslant u_{mn}[\psi(x,t)]$. Inequalities (29) are proved.

Equation (29) leads to the following similar inequalities for the inverse functions $\Phi[u_{mn}(\psi(x,t))] \equiv \sigma_{mn}(\psi)$, $\Phi[v_{mn}(x,t)] \equiv s_{mn}(x,t)$, and $\Phi[u_{mn}(x)] \equiv \sigma_{mn}(x)$:

$$\sigma_{mn}[\psi(x,t)] \leqslant s_{mn}(x,t) \leqslant \sigma_{mn}(x).$$

Since $|s(x,t) - \sigma(x)| \leqslant |s(x,t) - s_{mn}(x,t)| + |s_{mn}(x,t) - \sigma_{mn}(x)| + |\sigma_{mn}(x) - \sigma(x)|$, for any small ε there are numbers $N = N(\varepsilon)$ and $T = T(N)$ such that $|s(x,t) - \sigma(x)| < \varepsilon$ for $(m,n) > N$, $t > T$, i.e., as $t \to \infty$, $s(x,t)$ tends to the unique solution $\sigma(x)$ of problem (5), (6).

Next, taking into account that

$$|u_{mn}(x) - u_{mn}(\psi(x,t))| \leqslant M_0|x - \psi(x,t)| \leqslant \frac{M}{(t+1)^\gamma},$$

we come to the estimate

$$|v_{mn}(x,t) - u_{mn}(x)| \leqslant \frac{M}{(t+1)^\gamma}, \tag{30}$$

where $\gamma = \gamma(m,n,l,Q_0,\gamma_1)$; M depends on l, δ, and Q_0 and does not depend on m and n.

When inequalities (11) are satisfied, the constant γ can be chosen independently of m and n. Then, estimate (12) is obtained by passing to the limit $n \to \infty$ and then to $m \to \infty$ in inequalities (30). The theorem is proved.

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